ON THE INNER PARALLEL BODY OF A CONVEX BODY

BY

P. McMULLEN

ABSTRACT

It is shown that the set of boundary points of a convex body at which there are no interior tangent balls of positive radius has zero surface area.

In connexion with a problem concerning Minkowski decomposition of convex sets, Sallee [3] has asked the following question. Let K be a convex body (compact convex set with non-empty interior) in E^d, and let $B(x,\rho)$ denote the ball with centre x and radius ρ. If $y \in \mathrm{bd}K$, let

$$\rho(y) = \sup\{\rho \geqq 0 \mid y \in B(x,\rho) \subseteq K, \text{ for some } x\},$$

and for each $\epsilon \geqq 0$, let

$$K(\epsilon) = \{y \in \mathrm{bd}K \mid \rho(y) \leqq \epsilon\}.$$

Is it true that $K(0)$ has zero surface area? We shall answer this question in the affirmative.

We approach the question by means of the inner parallel bodies of K. If $\rho > 0$, the inner parallel body $K_{-\rho}$ of K is defined to be

$$K_{-\rho} = \{x \in E^d \mid B(x,\rho) \subseteq K\}.$$

Thus, if $y \in \mathrm{bd}K, \rho(y) > 0$ if and only if, for some $\rho > 0$, there is an $x \in \mathrm{bd}K_{-\rho}$ such that $y \in B(x,\rho)$. If $h(K,u)$ (where u is a unit vector) is the support function of K, we see that

$$K_{-\rho} = \{x \in E^d \mid \langle x, u\rangle \leqq h(K,u) - \rho, \text{ for all } u\}.$$

In particular, if $x \in \mathrm{bd}K_{-\rho}$, there is a $y \in \mathrm{bd}K$ such that $y \in B(x,\rho)$ and a unit vector u such that

Received March 18, 1974

$$\langle x, u\rangle = h(K_{-\rho}, u) = h(K, u) - \rho = \langle y, u\rangle - \rho.$$

It follows that x is the image of y under the nearest point map N_ρ; that is, $yN_\rho = x$ is the nearest point of $K_{-\rho}$ to y. We note [2, sec. 1.4] that the hyperplane through x with normal $y - x$ supports $K_{-\rho}$.

The points $x \in \mathrm{bd}K_{-\rho}$ fall into two classes, consisting of the regular and singular points. If x is a regular point, there is a unique support hyperplane to $K_{-\rho}$ at x; the corresponding point $y \in \mathrm{bd}K$ is then unique. If x is a singular point, there is more than one such support hyperplane, and so there may be more than one corresponding point y. (There will actually be more than one point y, but we shall not need this fact.)

Anderson-Klee [1] have shown that the set of singular points on the boundary of a convex body in E^d has σ-finite $(d-2)$-measure; in particular, it has zero surface area. That is, the set of regular points of $\mathrm{bd}K_{-\rho}$ has surface area $S(\mathrm{bd}K_{-\rho})$ (or $S(K_{-\rho})$, for brevity), the surface area of $K_{-\rho}$. Since the nearest point map N_ρ is distance decreasing ([2, §1.4]), it follows that

$$S(\mathrm{bd}K \setminus K(\rho)) \geqq S(K_{-\rho}).$$

Now, if $K_{-\rho} \neq \varnothing$ (which is true for all small enough ρ), there exists $\sigma > 0$, such that

$$K \subseteq (K_{-\rho})_\sigma = K_{-\rho} + B(0,\sigma),$$

and $\sigma \to 0$ as $\rho \to 0$. We then have

$$S(K) \leqq S((K_{-\rho})_\sigma) = S(K_{-\rho}) + O(\sigma).$$

Hence

$$S(K(\rho)) = S(K) - S(\mathrm{bd}K \setminus K(\rho)) \leqq S(K_{-\rho}) + O(\sigma) - S(K_{-\rho}) = O(\sigma).$$

In other words, $S(K(0)) = 0$, as we wished to show.

In view of the proof, it is natural to ask whether $K(0)$, too, must have σ-finite $(d-2)$-measure. However, this is not so; Professor C. A. Rogers has suggested the following counterexample K in E^2 (where σ-finite is equivalent to countable), for which $K(0)$ is a Cantor set. K is the convex hull of a Cantor set C on the unit circle; in constructing C by the usual process of successively deleting open intervals in the middle of each closed interval, we choose that, at the *n-th* stage of the construction, the ratio of the deleted interval to the whole interval shall be $1 - n^{-1}$. Then $K(0) = C$ will be uncountable. Taking the cartesian product of K with a unit $(d-2)$–cube (if $d \geqq 3$) gives us a counterexample in higher dimensions.

References

1. R. D. Anderson and V. L. Klee, *Convex functions and upper semi-continuous collections*, Duke Math. J. **19** (1952), 349–357.

2. P. McMullen and G. C. Shephard, *Convex polytopes and the upper bound conjecture*, London Math. Soc. Lecture Notes Series, Vol. 3, (Cambridge, 1971).

3. G. T. Sallee, *Minkowski decomposition of convex sets*, Israel J. Math. **12** (1972), 266–276.

Department of Mathematics
University College London
London, England